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# WHAT IF $\lambda$ IS A STRONG LIMIT SINGULAR CARDINAL ?

名古屋大学・人間情報学研究科 松原 洋 (YO MATSUBARA)

Nagoya University

## 1. BACKGROUND

Let $\kappa$ denote a regular uncountable cardinal and $\lambda$ a cardinal $\geq \kappa$. Let $\mathcal{P}_\kappa\lambda$ denote the set $\{x \subset \lambda \| x| < \kappa\}$. We refer the reader to Kanamori [6, Section 25] for basic facts about the combinatorics of $\mathcal{P}_\kappa\lambda$.

Suppose $I$ is an ideal over $\mathcal{P}_\kappa\lambda$. Let $I^+=\{X \subseteq \mathcal{P}_\kappa\lambda \mid X \notin I\}$. Let $\mathbb{P}_I$ denote the p.o. of members of $I^+$ ordered by $X \leq_{\mathbb{P}_I} Y \iff X \subseteq Y$.

**Definition 1.1.**
We say that an ideal $I$ is precipitous if $\Vdash_{\mathbb{P}_I}$ "Ult$(V;G)$ is wellfounded".

Let $NS_{\kappa\lambda}=\{X \subseteq \mathcal{P}_\kappa\lambda \mid X$ is the non-stationary$\}$. $NS_{\kappa\lambda}$ is known as the non-stationary ideal over $\mathcal{P}_\kappa\lambda$. For a stationary $X \subseteq \mathcal{P}_\kappa\lambda$, let $NS_{\kappa\lambda} \mid X$ denote the ideal over $\mathcal{P}_\kappa\lambda$ defined by $Y \in NS_{\kappa\lambda} \mid X \iff Y \cap X \in NS_{\kappa\lambda}$.

Can $NS_{\kappa\lambda}$ or $NS_{\kappa\lambda} \mid X$ be precipitous ?

**Answer.** : *Yes ( sometimes assuming ...).*

**Note** The existence of a precipitous ideal has the strength of some large cardinal because it provides us with a "*generic*" elementary embedding of **V**.

**Theorem 1.2 (Foreman, Magidor, Shelah, Goldring) [3][6].**
*If $\lambda$ is regular and $\delta$ is a Woodin cardinal $>\lambda$, then $\Vdash_{Coll(\lambda,<\delta)}$ "$NS_{\kappa\lambda}$ is precipitous". ($Coll(\lambda,<\delta)$ is the Levy collapse of $\delta$ to $\lambda^+$.)*

**Question.** *What if $\lambda$ is singular ?*

Burke and Matsubara [1] conjectured that $NS_{\kappa\lambda}$ cannot be precipitous if $\lambda$ is singular.

**Definition 1.3.** Let $\delta$ be a cardinal. We say that an ideal $I$ is *$\delta$-saturated* if $\mathbb{P}_I$ satisfies the $\delta$ chain condition .

**Fact.** If $I$ is a $\lambda^+$-saturated $\kappa$-complete normal ideal over $\mathcal{P}_\kappa\lambda$, then $I$ is precipitous.

**Note.** $NS_{\kappa\lambda}$ is the minimal $\kappa$-complete normal ideal over $\mathcal{P}_\kappa\lambda$.

**Theorem 1.4 (Foreman-Magidor) [2].**
*Unless $\kappa=\lambda=\aleph_1$, $NS_{\kappa\lambda}$ cannot be $\lambda^+$-saturated.*

What about $NS_{\kappa\lambda} \mid X$ ?

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**Menas' Conjecture.** *Every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

It turned out that Menas' Conjecture is independent of **ZFC**.

**Theorem 1.5.** $L \models$ *"Menas' Conjecture holds".*

**Theorem 1.6(Gitik) [5].** *Suppose that $\kappa$ is supercompact and $\lambda > \kappa$. Then $\exists$ p.o. $\mathbb{P}$ that preserves cardinals $\geq \kappa$ such that $\Vdash_{\mathbb{P}}$ "$\kappa$ is inaccessible and $\exists$ stationary $X \subseteq \mathcal{P}_\kappa\lambda$ such that $X$ cannot be partitioned into $\kappa^+$ disjoint stationary sets".*

## 2. Main results

**Theorem 2.1 (Matsubara-Shelah)[9].** *If $\lambda$ is a strong limit singular cardinal then $NS_{\kappa\lambda}$ is nowhere precipitous (i.e. $NS_{\kappa\lambda} \mid X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$).*

**Theorem 2.2 [9].** *If $\lambda$ is a strong limit singular cardinal then every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

One of the ingredients of the proof is the following lemma.

**Lemma 2.3.** *If $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$, then*

(i) *every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets and*
(ii) *$NS_{\kappa\lambda}$ is nowhere precipitous.(Matsubara-Shioya).*

*Remark.*

(1) The hypothesis of Lemma 2.3 is satisfied if $\lambda$ is a strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa$ .
(2) Under the hypothesis of Lemma 2.3, if $X \subseteq \mathcal{P}_\kappa\lambda$ has size $< 2^\lambda$ then $X$ is bounded and therefore non-stationary.

For the proof of (i) see page 345 of Kanamori [8].

*proof of* (ii).
Consider the following game $G_\omega$ between two players, **Nonempty** and **Empty**.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Nonempty** | $X_1$ | | $X_2$ | | $\dots$ | $X_n$ | | $\dots$ |
| **Empty** | | $Y_1$ | | $Y_2$ | $\dots$ | | $Y_n$ | $\dots$ |

**Nonempty** and **Empty** alternately choose stationary sets $X_n, Y_n \subseteq \mathcal{P}_\kappa\lambda$ respectively so that $X_n \supseteq Y_n \supseteq X_n$ for n=1,2,3,....

After $\omega$ moves, Empty wins $G_\omega$ if $\bigcap_{n=1}^{\infty} X_n = \emptyset$

**Fact.** $NS_{\kappa\lambda}$ is nowhere precipitous iff **Empty** has a winning strategy in $G_\omega$.

For the proof of this fact, see [4]. Let $\langle f_\alpha \mid \alpha < 2^\lambda \rangle$ enumerate functions from $\lambda^{<\omega}$ into $\mathcal{P}_\kappa\lambda$.
For a function $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$, let

$$\underbrace{C(f)}_{\text{club set generated by} f} = \{s \in \mathcal{P}_\kappa\lambda \mid \bigcup f"s^{<\omega} \subseteq s\}$$

**Fact.** $X \subseteq \mathcal{P}_\kappa\lambda$ is stationary iff $\forall\alpha < 2^\lambda\ C(f_\alpha) \cap X \neq \emptyset$.

We now describe **Empty**'s strategy. Suppose **Nonempty** plays $X_1$. Choose a sequence $\langle s^1_\alpha \mid \alpha < 2^\lambda\rangle$ from $X_1$ by induction on $\alpha$ as follow: Pick an element from $X_1 \cap C(f_0)$ and call it $s^1_0$.
Given $\langle s^1_\alpha \mid \alpha < \beta\rangle$ for some $\beta < 2^\lambda$, pick $s^1_\beta \in X_1 \cap C(f_\beta) \setminus \underbrace{\{s^1_\alpha \mid \alpha < \beta\}}_{\text{non-stationary}}$.

Let **Empty** play $Y_1 = \{s^1_\alpha \mid \alpha < 2^\lambda\}$. Now suppose **Nonempty** plays $X_n$ immediately following **Empty**'s move $Y_{n-1} = \{s^{n-1}_\alpha \mid \alpha < 2^\lambda\}$.
Choose $\langle s^n_\alpha \mid \alpha < 2^\lambda\rangle$ a sequence from $X_n$ as follows:
Pick $s^n_0 \in (X \cap C(f_\beta)) \setminus \underbrace{(\{s^{n-1}_\alpha \mid \alpha \leq \beta\} \cup \{s^n_\alpha \mid \alpha < \beta\})}_{\text{non-stationary}}$.

Let **Empty** play $Y_n = \{s^n_\alpha \mid \alpha < 2^\lambda\}$.

**Claim.** *This is a winning strategy for* **Empty**

*proof*: We want to show that $\bigcap\limits_{n=1}^{\infty} Y_n = \emptyset$.

Suppose otherwise, say $t \in \bigcap\limits_{n=1}^{\infty} Y_n$. For each $n < \omega$, $\exists\,!\ \alpha_n < 2^\lambda$ such that $t = s^n_{\alpha_n}$.
It is easy to see that $\alpha_n > \alpha_{n+1}$ for each $n$. ($s^n_\beta \notin \{s^n_\alpha \mid \alpha \leq \beta\}$ etc ...)

We now prove Theorem 2.2 assuming Theorem 2.1 and Lemma 2.3 (i).

*proof of Theorem 2.2.* : Let $\lambda$ be a strong limit singular cardinal . If $\mathrm{cf}(\lambda) < \kappa$ then by Lemma 2.3 (i) we are done.

Assume $\mathrm{cf}(\lambda) \geq \kappa$. In this case $\lambda^{<\kappa} = \lambda$. So it is enough to show that $NS_{\kappa\lambda} \mid X$ is not $\lambda$-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$.

But this is a consequence of $NS_{\kappa\lambda}$ being nowhere precipitous . In fact we know that $NS_{\kappa\lambda} \mid X$ cannot be $\lambda^+$-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$.

*proof of Theorem 2.1.* : We now tamper with the definition of $\mathcal{P}_\kappa\lambda$.
From now on we let $\mathcal{P}_\kappa\lambda = \{s \subseteq \lambda \mid |s| < \kappa, s \cap \kappa \in \kappa\}$. This set is club in $\{s \subseteq \lambda \mid |s| < \kappa\}$. The following is the advantage of this change:
$X \subseteq \mathcal{P}_\kappa\lambda$ is stationary iff $\forall f : \lambda^{<\omega} \to \lambda \quad C[f] \cap X \neq \emptyset$
where $C[f] = \{s \in \mathcal{P}_\kappa\lambda \mid s \text{ is closed under} f\}$.

Let $\lambda$ be a strong limit singular cardinal. By Lemma 2.3 (ii) we may assume that $\mathrm{cf}(\lambda) \geq \kappa$. Let $\langle\lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ be a continuous increasing sequence of strong limit singular cardinals converging to $\lambda$. Let $T = \{i < \mathrm{cf}(\lambda) \mid \mathrm{cf}(i) < \kappa\}$.
For each $i \in T$, let $E_i = \{s \in \mathcal{P}_\kappa\lambda \mid \sup(s) = \lambda_i, \lambda_i \notin s\}$

**Note.**

(i) $|E_i| = 2^{\lambda_i}$
(ii) $\bigcup\limits_{i \in T} E_i$ is club in $\mathcal{P}_\kappa\lambda$.

For each $i \in T$, let $\langle f^i_\epsilon \mid \epsilon < 2^{\lambda_i}\rangle$ enumerate all of the functions whose domain $\subseteq \lambda_i^{<\omega}$ and range $\subseteq \lambda_i$.

**Definition 2.4.** $C^i[f^i_\epsilon] = \{s \in E_i \mid s^{<\omega} \subseteq dom(f^i_\epsilon)$ and $s$ is close

To show $NS_{\kappa\lambda}$ is nowhere precipitous we will present a win **Empty** in $G_\omega$.
Suppose $W_1$ is **Nonempty**'s first move in $G_\omega$. For each $i \in T$, we a "*local game*" where each player altenately chooses subsets of $E_i$ **Nonempty**'s first move is $W_1 \cap E_i$.
**Local game** $G(i)$
For each $i \in T$, define a game $G(i)$ as follows:
**Nonempty** and **Empty** alternately choose $X_n, Y_n \subseteq E_i$ resp $1, 2, \ldots,$ so that $X_n \supseteq Y_n \supseteq X_{n+1}$ and $\forall \epsilon < 2^{\lambda_i}$ ( $|C^i[f^i_\epsilon] \cap$ $C^i[f^i_\epsilon] \cap Y_n \neq \emptyset$ ).
**Empty** wins $G(i)$ iff $\bigcap_{n=1}^{\infty} X_n = \emptyset$.
Just as in the proof of Lemma 2.3 (ii) we can show that **Empt** strategy, say $\tau_i$ in $G_i$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $G_\omega$ | **Nonempty** | $W_1$ | | | $W_2$ | |
| | **Empty** | $\downarrow$ | $\bigcup_{i \in T} \tau_i(\langle W \cup E_i \rangle)$ | | $\downarrow$ | $\bigcup_{i \in}$ |
| $G(i)$ | | $W_1 \cap E_i$ | $\uparrow$ | | $W_2 \cap E_i$ | |
| $(i \in T)$ | | | $\tau_i(\langle W \cap E_i \rangle)$ | | | |

The following lemma tells us that we can combine $\tau_i$'s for $i \in T$ for $G_\omega$.

**Lemma 2.5.** *Suppose $W \subseteq \mathcal{P}_\kappa\lambda$ is stationary. If $U \subseteq \mathcal{P}_\kappa\lambda$ satis condition (♯) then $U$ is stationary.*
*(♯) For each $i \in T$, $\forall \epsilon < 2^{\lambda_i}$( $|C^i[f^i_\epsilon] \cap W| = 2^{\lambda_i} \longrightarrow C^i[f^i_\epsilon] \cap U \neq$*

Now we describe **Empty**'s (combined) strategy $\sigma$ in $G_\omega$. Supp plays $W_1$.
Let **Empty** play $\bigcup_{i \in T} \tau_i(\langle W_1 \cap E_i \rangle) \stackrel{def}{=} \sigma(\langle W_1 \rangle)$.
Suppose

$$\begin{array}{ccccc} W_1 & & W_2 & & \ldots \quad W_n \\ & \sigma(\langle W_1 \rangle) & & \sigma(\langle W_1, W_2 \rangle) & \ldots \end{array}$$

is the run of the game $G_\omega$ so far.
Let

$$\sigma(\langle W_1, W_2, \ldots, W_n \rangle) \stackrel{def}{=} \bigcup_{i \in T} \tau_i(\langle W_1 \cap E_i, W_2 \cap E_i, \ldots, W_n$$

Lemma 2.5 guarantees that $\sigma$ provides **Empty** a legal move i.e. st of **Nonempty**'s last move. This $\sigma$ is a winning strategy for **Emp** The proof of Lemma 2.5 depends upon the following lemma whos theory.

**Lemma 2.6.** *Suppose $U \subseteq \mathcal{P}_\kappa\lambda$. If $\forall i \in T$ $|U \cap E_i| < 2^{\lambda_i}$, then $U$ is*

To prove the last lemma, we need the following fact from pcf theo pcf **Fact.** *$\exists$ club $C \subseteq cf(\lambda)$ such that $pp(\lambda_i) = 2^{\lambda_i}$ for every $i \in C$.*

See Shelah "Cardinal Arithmetic" [12] Conclusion 5.13 page 414 and Hotz,Steffens,Weitz "Introduction to Cardinal Arithmetic" [7] Theorem 9.1.3 page 271.

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